29　定積分と級数

基本問題 & 解法のポイント

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解法例 1

曲線の閉区間$[0,\ \pi]$をn等分してできる不連続なn個の長方形の面積和の極限と見なすと,

$$\begin{aligned}\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n}\cos^2\left\{\frac{1}{6}\cdot\left(\frac{\pi}{n}\cdot k\right)\right\}&=\int_0^\pi\cos^2\frac{1}{6}xdx\\&=\frac{1}{2}\int_0^\pi\left(1+\cos\frac{1}{3}x\right)dx\\&=\frac{1}{2}\left[x+3\sin\frac{1}{3}x\right]_0^\pi\\&=\frac{\pi}{2}+\frac{3\sqrt{3}}{4}\end{aligned}$$

解法例 2

曲線の閉区間$[0,\ 1]$をn等分してできる不連続なn個の長方形の面積和の極限と見なすと,

$$\begin{aligned}\lim_{n\to\infty}\frac{\pi}{n}\sum_{k=1}^{n}\cos^2\frac{k\pi}{6n}&=\pi\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\cos^2\left(\frac{\pi}{6}\cdot\frac{k}{n}\right)\\&=\pi\int_0^1\cos^2\frac{\pi}{6}xdx\\&=\frac{\pi}{2}\int_0^1\left(1+\cos\frac{\pi}{3}x\right)dx\\&=\frac{\pi}{2}\left[x+\frac{3}{\pi}\sin\frac{\pi}{3}x\right]_0^1\\&=\frac{\pi}{2}+\frac{3\sqrt{3}}{4}\end{aligned}$$

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曲線の閉区間$[0,\ 1]$をn等分してできる不連続なn個の長方形の面積和の極限と見なすと,

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{2k}{n^2+k^2}&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{2k}{n+\dfrac{k^2}{n}}\\&=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{2\left(\dfrac{k}{n}\right)}{1+\left(\dfrac{k}{n}\right)^2}\\&=\int_0^1\frac{2x}{1+x^2}dx\\&=\left[\log\left(1+x^2\right)\right]_0^1\\&=\log 2\end{aligned}$$

A

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(1)

$$\lim_{n\to\infty}\frac{(1+2+3+\cdots+n)^5}{(1^4+2^4+3^4+\cdots+n^4)^2}=\lim_{n\to\infty}\frac{\left(\sum_{k=1}^{n}k\right)^5}{\left(\sum_{k=1}^{n}k^4\right)^2}$$

$$=\lim_{n\to\infty}\frac{\left(n\sum_{k=1}^{n}\frac{k}{n}\right)^5}{\left\{n^4\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}^2}$$

$$=\lim_{n\to\infty}\frac{\left(\frac{n^2}{n}\sum_{k=1}^{n}\frac{k}{n}\right)^5}{\left\{\frac{n^5}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}^2}$$

$$=\lim_{n\to\infty}\frac{n^{10}\left(\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\right)^5}{n^{10}\left\{\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}^2}$$

$$=\frac{\left(\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{k}{n}\right)^5}{\left\{\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left(\frac{k}{n}\right)^4\right\}^2}$$

$$=\frac{\left(\int_0^1 xdx\right)^5}{\left(\int_0^1 x^4dx\right)^2}$$

$$=\frac{\left(\left[\frac{x^2}{2}\right]_0^1\right)^5}{\left(\left[\frac{x^5}{5}\right]_0^1\right)^2}$$

$$=\frac{25}{32}$$

(2)

$$\log\frac{1}{n}\sqrt[n]{\frac{(4n)!}{(3n)!}}=\log\left[\frac{4n\cdot(4n-1)\cdot(4n-2)\cdot\cdots\cdot\{4n-(n-1)\}}{n^n}\right]^{\frac{1}{n}}$$
$$=\frac{1}{n}\log\left\{4\cdot\left(4-\frac{1}{n}\right)\cdot\left(4-\frac{2}{n}\right)\cdot\cdots\cdot\left(4-\frac{n-1}{n}\right)\right\}$$
$$=\frac{1}{n}\left\{\log 4+\log\left(4-\frac{1}{n}\right)+\log\left(4-\frac{2}{n}\right)+\cdots+\log\left(4-\frac{n-1}{n}\right)\right\}$$
$$=\frac{1}{n}\sum_{k=0}^{n-1}\log\left(4-\frac{k}{n}\right)$$

$$\log\left\{\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(4n)!}{(3n)!}}\right\}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=0}^{n-1}\log\left(4-\frac{k}{n}\right)$$
$$=\int_0^1\log(4-x)dx$$

ここで，$4-x=t$ とおくと，$dx=-dt,\ x=1\Leftrightarrow t=3,\ x=0\Leftrightarrow t=4$ より，

$$\log\left\{\lim_{n\to\infty}\frac{1}{n}\sqrt[n]{\frac{(4n)!}{(3n)!}}\right\}=\int_0^1\log(4-x)dx$$
$$=-\int_4^3\log t\,dt$$
$$=\int_3^4\log t\,dt$$
$$=\left[t\log t-t\right]_3^4$$
$$=4\log 4-3\log 3-1$$
$$=\log\frac{256}{27e}$$

よって，$\frac{1}{n}\sqrt[n]{\frac{(4n)!}{(3n)!}}=\frac{256}{27e}$

(3)

$$\frac{\pi}{n}\left\{\frac{1}{\sin\frac{\pi(n+1)}{4n}}+\frac{1}{\sin\frac{\pi(n+2)}{4n}}+\cdots+\frac{1}{\sin\frac{\pi(n+n)}{4n}}\right\}=\frac{\pi}{n}\sum_{k=1}^{n}\frac{1}{\sin\frac{\pi(n+k)}{4n}}$$

$$=\frac{1}{n}\sum_{k=1}^{n}\frac{\pi}{\sin\frac{\pi}{4}\left(1+\frac{k}{n}\right)}$$

より，

$$\lim_{n\to\infty}\frac{\pi}{n}\left\{\frac{1}{\sin\frac{\pi(n+1)}{4n}}+\frac{1}{\sin\frac{\pi(n+2)}{4n}}+\cdots+\frac{1}{\sin\frac{\pi(n+n)}{4n}}\right\}=\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\frac{\pi}{\sin\frac{\pi}{4}\left(1+\frac{k}{n}\right)}$$

$$=\int_0^1\frac{\pi}{\sin\frac{\pi(1+x)}{4}}dx$$

ここで，$\frac{\pi(1+x)}{4}=t$ とおくと，$dx=\frac{4}{\pi}dt,\ x=1\Leftrightarrow t=\frac{\pi}{2},\ x=0\Leftrightarrow t=\frac{\pi}{4}$

ゆえに,

$$\lim_{n\to\infty}\frac{\pi}{n}\left\{\frac{1}{\sin\frac{\pi(n+1)}{4n}}+\frac{1}{\sin\frac{\pi(n+2)}{4n}}+\cdots+\frac{1}{\sin\frac{\pi(n+n)}{4n}}\right\}=\int_0^1\frac{\pi}{\sin\frac{\pi(1+x)}{4}}dx$$

$$=\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\pi}{\sin t}\cdot\frac{4}{\pi}dt$$

$$=4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{1}{\sin t}dt$$

$$=4\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\frac{\sin t}{(1-\cos t)(1+\cos t)}dt$$

$$=2\int_{\frac{\pi}{4}}^{\frac{\pi}{2}}\left(\frac{\sin t}{1-\cos t}+\frac{\sin t}{1+\cos t}\right)dt$$

$$=2\Big[\log(1-\cos t)-\log(1+\cos t)\Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}}$$

$$=2\left\{-\log\left(1-\frac{1}{\sqrt{2}}\right)+\log\left(1+\frac{1}{\sqrt{2}}\right)\right\}$$

$$=4\log\left(\sqrt{2}+1\right)$$

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(1)

ABの中点をOとすると,

$$\angle \mathrm{BAP}_k = \frac{\pi - \angle \mathrm{AOP}_k}{2} = \frac{\pi - \frac{k}{n}\pi}{2} = \frac{\pi}{2} - \frac{k}{2n}\pi$$

より,

$$\mathrm{AP}_k + \mathrm{P}_k\mathrm{B} + \mathrm{BA} = \mathrm{AB}\cos\left(\frac{\pi}{2} - \frac{k}{2n}\pi\right) + \mathrm{AB}\sin\left(\frac{\pi}{2} - \frac{k}{2n}\pi\right) + \mathrm{BA} = 2\left(\sin\frac{k}{2n}\pi + \cos\frac{k}{2n}\pi + 1\right)$$

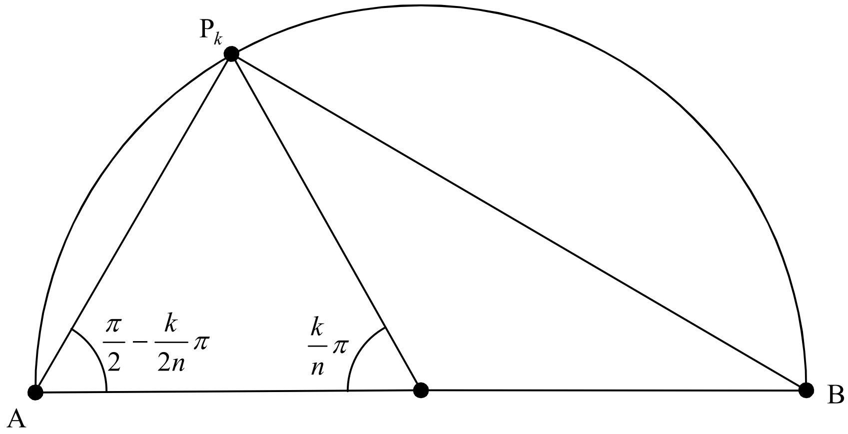
よって, $l_n(k) = 2\left(\sin\frac{k}{2n}\pi + \cos\frac{k}{2n}\pi + 1\right)$

(2)

$$\frac{l_n(1) + l_n(2) + \cdots + l_n(n)}{n} = \frac{1}{n}\sum_{k=1}^{n} l_n(k) = \frac{1}{n}\sum_{k=1}^{n} 2\left\{\sin\left(\frac{\pi}{2}\cdot\frac{k}{n}\right) + \cos\left(\frac{\pi}{2}\cdot\frac{k}{n}\right) + 1\right\}$$

よって,

$$\lim_{n\to\infty}\frac{l_n(1) + l_n(2) + \cdots + l_n(n)}{n} = 2\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\left\{\sin\left(\frac{\pi}{2}\cdot\frac{k}{n}\right) + \cos\left(\frac{\pi}{2}\cdot\frac{k}{n}\right) + 1\right\}$$
$$= 2\int_0^1\left(\sin\frac{\pi}{2}x + \cos\frac{\pi}{2}x + 1\right)dx$$
$$= 2\left[-\frac{2}{\pi}\cos\frac{\pi}{2}x + \frac{2}{\pi}\sin\frac{\pi}{2}x + x\right]_0^1$$
$$= 2\left(\frac{4}{\pi} + 1\right)$$



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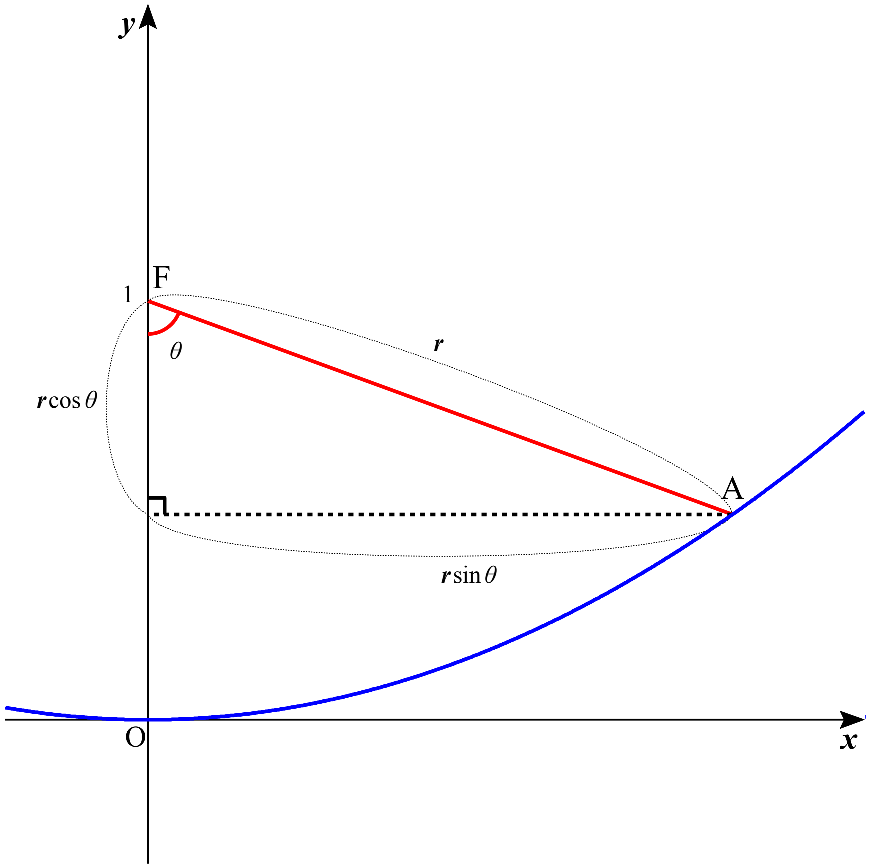
(1)

$\mathrm{A}(x,\ y)=(r\sin\theta,\ 1-r\cos\theta)$

点 A は $y=\frac{1}{4}x^2$　すなわち　$x^2-4y=0$ を満たすことから，$\sin^2\theta\cdot r^2+4\cos\theta\cdot r-4=0$

これと，$\sin\theta\neq0\ (\because x>0)$，$r>0$ より，

$$r=\frac{-2\cos\theta+\sqrt{4\cos^2\theta+4\sin^2\theta}}{\sin^2\theta}=\frac{-2\cos\theta+2}{1-\cos^2\theta}=\frac{2(1-\cos\theta)}{(1-\cos\theta)(1+\cos\theta)}=\frac{2}{1+\cos\theta}$$



(2)

$$\lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\mathrm{FA}_k=\lim_{n\to\infty}\frac{1}{n}\sum\frac{2}{1+\cos\frac{k\pi}{2n}}$$

$$=\frac{2}{\pi}\lim_{n\to\infty}\frac{\frac{\pi}{2}}{n}\sum_{k=1}^{n}\frac{2}{1+\cos\frac{\frac{\pi}{2}}{n}k}$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{2}{1+\cos x}dx$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{2}{2\cos^2\frac{x}{2}}dx$$

$$=\frac{2}{\pi}\int_0^{\frac{\pi}{2}}\frac{dx}{\cos^2\frac{x}{2}}dx$$

$$=\frac{2}{\pi}\left[2\tan\frac{x}{2}\right]_0^{\frac{\pi}{2}}$$

$$=\frac{4}{\pi}$$

B

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三角錐$\mathrm{OP}_k\mathrm{P}_{k+1}\mathrm{Q}_k$の底面を三角形$\mathrm{OP}_k\mathrm{P}_{k+1}$とすると高さは$\mathrm{OQ}_k$である。

三角形$\mathrm{OP}_k\mathrm{P}_{k+1}$の面積

条件より点P_kは$y=1-x$（$0\leq x\leq 1$）を満たすxy平面上の線分（長さ$\sqrt{2}$）をn等分し，

端点$(0,\ 1)$から順にP_0，P_1，P_2，P_nとした点であるから，$\mathrm{P}_k\mathrm{P}_{k+1}=\dfrac{\sqrt{2}}{n}$

また，O と$y=1-x$の距離は$\dfrac{1}{\sqrt{2}}$

よって，三角形$\mathrm{OP}_k\mathrm{P}_{k+1}$の面積は$\dfrac{1}{2}\cdot\dfrac{\sqrt{2}}{n}\cdot\dfrac{1}{\sqrt{2}}=\dfrac{1}{2n}$

OQ_kの長さ

三角形$\mathrm{OP}_k\mathrm{Q}_k$は$\mathrm{P}_k\mathrm{Q}_k=1$，$\angle\mathrm{O}=90°$の直角三角形だから，

三平方の定理より，

$$\begin{aligned}\mathrm{OQ}_k&=\sqrt{1-\mathrm{OP}_k{}^2}\\&=\sqrt{1-\left\{\left(\frac{k}{n}\right)^2+\left(1-\frac{k}{n}\right)^2\right\}}\\&=\sqrt{2}\sqrt{\frac{k}{n}-\left(\frac{k}{n}\right)^2}\end{aligned}$$

よって，三角錐$\mathrm{OP}_k\mathrm{P}_{k+1}\mathrm{Q}_k$の体積$V_k$は

$$\begin{aligned}V_k&=\frac{1}{3}\cdot\frac{1}{2n}\cdot\sqrt{2}\sqrt{\frac{k}{n}-\left(\frac{k}{n}\right)^2}\\&=\frac{\sqrt{2}}{6}\cdot\frac{1}{n}\sqrt{\frac{k}{n}-\left(\frac{k}{n}\right)^2}\end{aligned}$$

ゆえに，

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}V_k&=\frac{\sqrt{2}}{6}\lim_{n\to\infty}\frac{1}{n}\sqrt{\frac{k}{n}-\left(\frac{k}{n}\right)^2}\\&=\frac{\sqrt{2}}{6}\int_0^1\sqrt{x-x^2}\,dx\end{aligned}$$

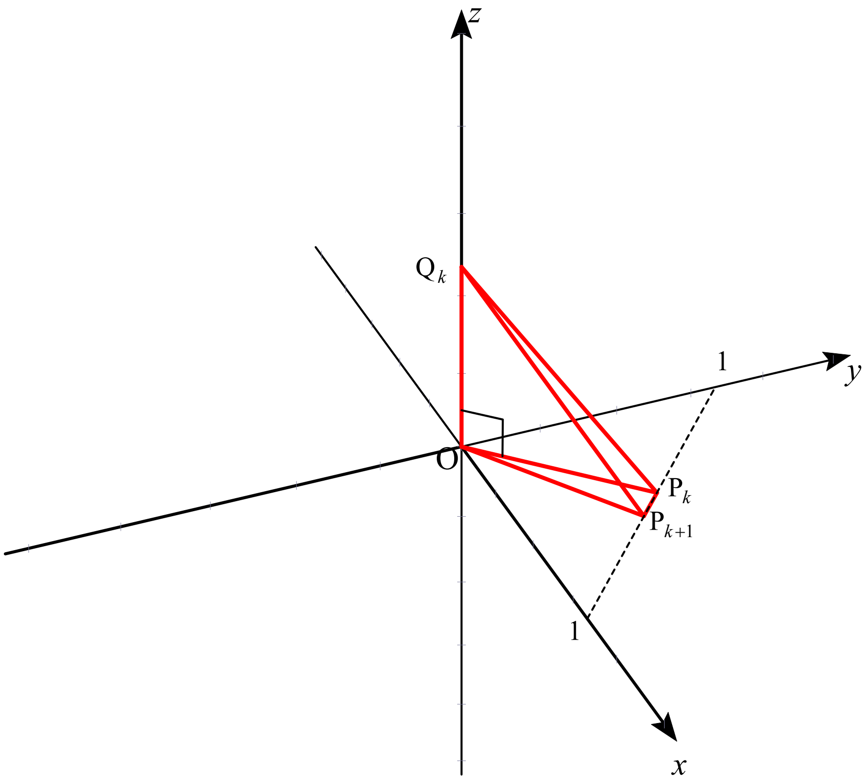
ここで，$y=\sqrt{x-x^2}$（$0\leq x\leq 1$）とおくと，

$y^2=x-x^2$（$0\leq x\leq 1,\ y\geq 0$）より，$\left(x-\dfrac{1}{2}\right)^2+y^2=\left(\dfrac{1}{2}\right)^2$（$0\leq x\leq 1,\ y\geq 0$）

したがって，$\int_0^1 \sqrt{x-x^2}\,dx$ は中心 $\left(\frac{1}{2},\ 0\right)$，半径 $\frac{1}{2}$ の円の上半分の面積を表す。

$$\therefore \lim_{n\to\infty}\sum_{k=1}^{n} V_k = \frac{\sqrt{2}}{6}\cdot\frac{1}{2}\pi\left(\frac{1}{2}\right)^2$$

$$= \frac{\sqrt{2}}{48}\pi$$



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$[x]$の定義より，$\left[\sqrt{2n^2-k^2}\right] \leq \sqrt{2n^2-k^2} < \left[\sqrt{2n^2-k^2}\right]+1$

すなわち　$\sqrt{2n^2-k^2}-1 < \left[\sqrt{2n^2-k^2}\right] \leq \sqrt{2n^2-k^2}$

よって，$\displaystyle\sum_{k=1}^{n}\frac{\sqrt{2n^2-k^2}-1}{n^2} < \sum_{k=1}^{n}\frac{\left[2n^2-k^2\right]}{n^2} \leq \sum_{k=1}^{n}\frac{\sqrt{2n^2-k^2}}{n^2}$

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{2n^2-k^2}}{n^2} &= \lim_{n\to\infty}\frac{1}{n}\sum_{k=1}^{n}\sqrt{2-\left(\frac{k}{n}\right)^2} \\ &= \int_0^1\sqrt{2-x^2}\,dx \\ &= \sqrt{2}\int_0^1\sqrt{1-\left(\frac{x}{\sqrt{2}}\right)^2}\,dx\end{aligned}$$

ここで$\dfrac{x}{\sqrt{2}}=\sin\theta$ とおくと，$dx=\sqrt{2}\cos\theta d\theta,\ x=1 \Leftrightarrow \theta=\dfrac{\pi}{4},\ x=0 \Leftrightarrow \theta=0$ より，

$$\begin{aligned}\int_0^1\sqrt{2-x^2}\,dx &= \sqrt{2}\int_0^{\frac{\pi}{4}}\sqrt{1-\sin^2\theta}\cos\theta\,d\theta \\ &= \sqrt{2}\int_0^{\frac{\pi}{4}}\cos^2\theta\,d\theta \\ &= \frac{\sqrt{2}}{2}\int_0^{\frac{\pi}{4}}(1+\cos 2\theta)d\theta \\ &= \frac{\sqrt{2}}{2}\left[\theta+\frac{\sin 2\theta}{2}\right]_0^{\frac{\pi}{4}} \\ &= \frac{\sqrt{2}}{2}\left(\frac{\pi}{4}+\frac{1}{2}\right)\end{aligned}$$

よって，$\displaystyle\lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{2n^2-k^2}}{n^2}=\frac{\pi}{4}+\frac{1}{2}$

また，これより，

$$\begin{aligned}\lim_{n\to\infty}\sum_{k=1}^{n}\left(\frac{\sqrt{2n^2-k^2}-1}{n^2}\right) &= \lim_{n\to\infty}\sum_{k=1}^{n}\frac{\sqrt{2n^2-k^2}}{n^2}-\lim_{n\to\infty}\sum_{k=1}^{n}\frac{1}{n^2} \\ &= \frac{\pi}{4}+\frac{1}{2}-\lim_{n\to\infty}\frac{1}{n} \\ &= \frac{\pi}{4}+\frac{1}{2}\end{aligned}$$

ゆえに，はさみうちの原理により，$\displaystyle\lim_{n\to\infty}a_n=\frac{\pi}{4}+\frac{1}{2}$

補足

$\int_0^1 \sqrt{2-x^2}\,dx$ について

$y=\sqrt{2-x^2}$ （$0 \leq x \leq 1$） $\Leftrightarrow$ $x^2+y^2=2$ （$0 \leq x \leq 1,\ y \geq 0$） より，

$\int_0^1 \sqrt{2-x^2}\,dx$ は下図斜線部の面積を表す。

よって，$\int_0^1 \sqrt{2-x^2}\,dx = \frac{1}{2}\cdot\left(\sqrt{2}\right)^2\cdot\frac{\pi}{4}+\frac{1}{2}\cdot 1\cdot 1 = \frac{\pi}{4}+\frac{1}{2}$